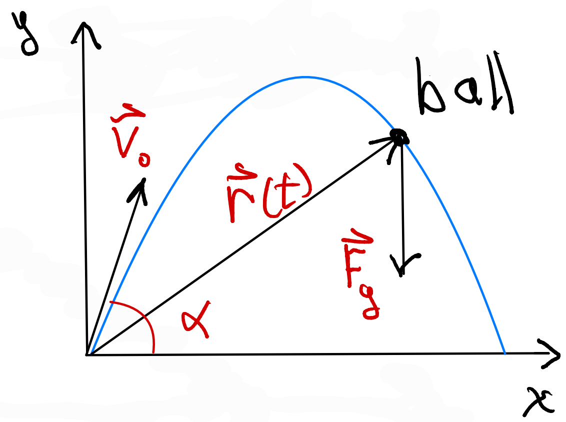
§ 13.2 Ideal Projectile Motion

- Assume a cannonball of mass M is fired upward @ angle α, velocity v_0. Find the maximum ht & distance traveled as a function of initial speed v_0 and angle α
- Find the angle α which shoots the ball furthest



$\boxed{\text{Setup}}$

- Force of gravity: $\vec{F}_g = \overrightarrow{(0,-mg)} = m\overrightarrow{(0,-g)}$

Not inverse square because we are taking an approximation that only holds near surface of the earth

- Newton's Law: $m\vec{a} = \vec{F}_g$

$$m\ddot{\vec{r}} = \vec{F}_g$$

- $\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j}$

$\dot{\vec{r}}(t) = \dot{x}(t)\,\underset{\sim}{i} + \dot{y}(t)\,\underset{\sim}{j} = \vec{v}(t)$

$\ddot{\vec{r}}(t) = \ddot{x}(t)\,\underset{\sim}{i} + \ddot{y}(t)\,\underset{\sim}{j} = \vec{a}(t)$

- **Note:** In the Kepler problem, we were given the motions of the planets, and set out to deduce the acceleration vector for the force that was creating that motion –

- In most applications of Newton's Theory, the force is given as a function of $\vec{r}$, and $\vec{F} = m\vec{a} = m\ddot{\vec{r}}$ then becomes a 2nd order ordinary differential equation for the motion $\vec{r}(t)$. This requires two initial conditions $\vec{r}_0 = \vec{r}(t_0)$, $\vec{v}_0 = \vec{r}\,'(t_0)$, imposed at t_0 $(t_0 = 0)$

- For example: Starting with the assumption of Newton's Force Law $\vec{F} = \cancel{M_p}\vec{a} = -G\frac{\cancel{M_p}M_s}{r^2}\frac{\vec{r}}{r}$ we could prove orbits are ellipses by solving the ODE $\boxed{\vec{r}\,''(t) = -G\frac{M_s}{r^3}\vec{r}}$

Much harder to solve!!

• Equations: $m\overrightarrow{(\ddot{x},\ddot{y})} = \overrightarrow{(0,-mg)} = m\overrightarrow{(0,-g)}$

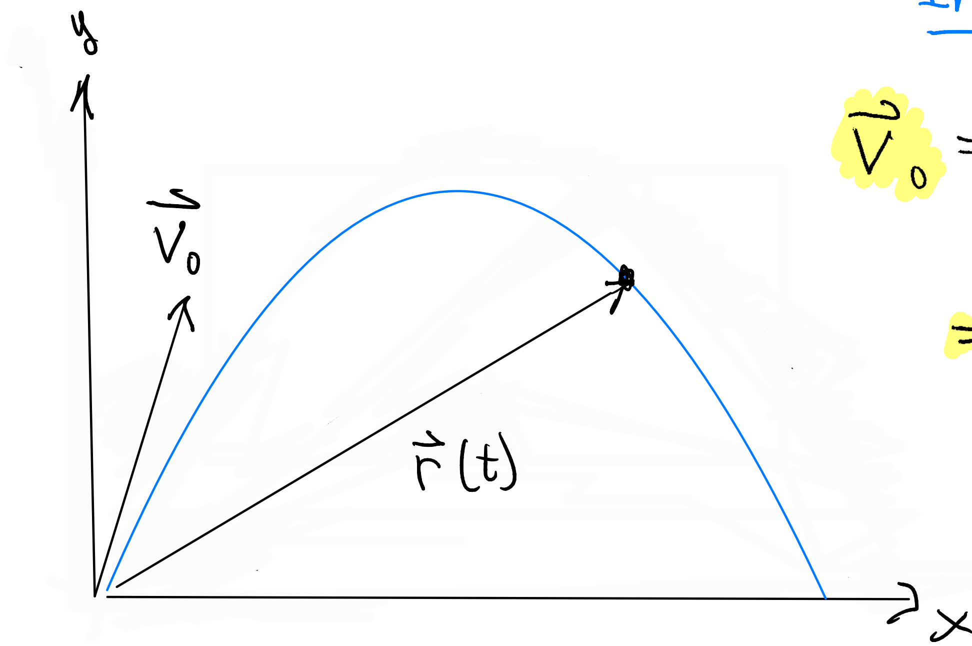
(for the trajectory $\vec{r}(t)$)

$$\overrightarrow{(\ddot{x},\ddot{y})} = \overrightarrow{(0,-g)}$$

$$\ddot{x} = 0$$
$$\ddot{y} = -g$$

"The equations of motion"

Picture:



Initial Conditions

$$\vec{V}_0 = \overrightarrow{(V_x^0, V_y^0)}$$
$$= V_0 \overrightarrow{(\cos\alpha, \sin\alpha)}$$

speed $= \|\vec{V}_0\| = V_0$

• Integrate:

$\ddot{x} = 0$ (x-ode)

$$\dot{x} = V_x^0$$
$$\boxed{x(t) = V_x^0 t + x_0}$$

$\ddot{y} = -g$ (y-ode)

$$\dot{y} = -gt + V_y^0$$
$$\boxed{y(t) = -\frac{1}{2}gt^2 + V_y^0 t + y_0}$$

We can solve the ODE's by direct integration

Simplest case!

- Put in initial conditions —

$\vec{r}(0) = \overrightarrow{(x_0, y_0)} = \overrightarrow{(0,0)}$ [cannonball on ground at $x=0, y=0$ @ start]

$\vec{v}(0) = \overrightarrow{(V_x^0, V_y^0)} = \overrightarrow{V_0(\cos\alpha, \sin\alpha)}$

$\vec{r}(t) = (V_0 \cos\alpha)t\, \underset{\sim}{i} + \left(-\frac{1}{2}gt^2 + (V_0 \sin\alpha)t\right) \underset{\sim}{j}$

Solution: $\begin{cases} x(t) = (V_0 \cos\alpha)t \\ y(t) = -\frac{1}{2}gt^2 + (V_0 \sin\alpha)t \end{cases}$

Find Max

- $y(t)$ maximized when $\dot{y}(t) = 0$

$$\dot{y} = -gt + V_0 \sin\alpha = 0$$

$$\boxed{t_{max} = \frac{V_0}{g} \sin\alpha}$$

Gives time t_{max} at which max ht is achieved

$y(t) = 0$ when $-\frac{1}{2}gt^2 + V_0 \sin\alpha\, t = 0$

$$t\left(-\frac{1}{2}gt + V_0 \sin\alpha\right) = 0$$

$\boxed{t = 2\frac{V_0}{g}\sin\alpha = 2t_{max}}$ ball hits ground

- Max height:

$$y(t_{max}) = -\frac{1}{2}g\Big(\underbrace{\frac{V_0}{g}\sin\alpha}_{t_{max}}\Big)^2 + (V_0\sin\alpha)\underbrace{\frac{V_0}{g}\sin\alpha}_{t_{max}}$$

$$= -\frac{1}{2}\frac{V_0^2}{g}\sin^2\alpha + \frac{V_0^2}{g}\sin^2\alpha$$

$$\boxed{y_{max} = \frac{1}{2}\frac{V_0^2}{g}\sin^2\alpha}$$

- Distance traveled:

$$x(2t_{max}) = (V_0\cos\alpha)\Big(\underbrace{\frac{2V_0}{g}\sin\alpha}_{2t_{max}}\Big)$$

$$= \frac{2V_0^2}{g}\cos\alpha\sin\alpha = \frac{V_0^2}{g}\sin 2\alpha$$

$$\boxed{x_{max} = \frac{V_0^2}{g}\sin 2\alpha}$$

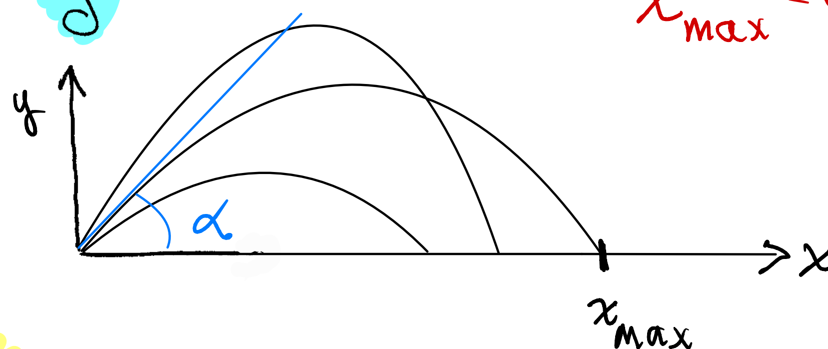
Example

- At a given velocity v_0, what angle maximizes distance?

Soln: $x_{max} = \frac{v_0^2}{g}\sin 2\alpha$

Find α such that $x_{max} = 0$

$0 \le \alpha \le \frac{\pi}{2}$



$\Rightarrow \sin 2\alpha = 0 = \sin(2 \cdot \frac{\pi}{4})$

max value of $\sin 2\alpha$ is at $\alpha = \frac{\pi}{4}$, $\sin(2 \cdot \frac{\pi}{4}) = 1$

Conclude: optimal angle is $\frac{\pi}{4}$ $(45°)$

Recall the

• Liebniz Rule: Dot product

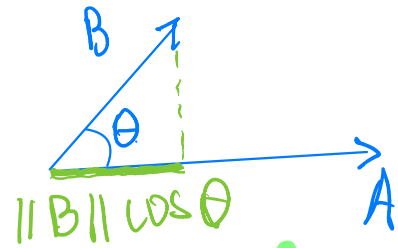
$$A \cdot B = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Multiply corresponding entries & add

$$= \|A\| \, \|B\| \cos\theta .$$

geometrical interpretation



Liebniz Product Rule –

$$\frac{d}{dt}\left(\vec{r}_1(t) \cdot \vec{r}_2(t)\right) = \dot{\vec{r}}_1(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \dot{\vec{r}}_2(t).$$

(Same for cross product – "That is why they are called products")

Application – Thm: if $\|\vec{r}(t)\| = \text{const}$, then $\vec{v}(t) \perp \vec{r}(t)$. & if $\|\vec{v}(t)\| = \text{const}$, then $\vec{a}(t) \perp \vec{v}(t)$.

Proof: $\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{d}{dt}\|\vec{r}\|^2 = 0$ Also

$$\frac{d}{dt}(\vec{r}\cdot\vec{r}) = 2\dot{\vec{r}}\cdot\vec{r} \Rightarrow \dot{\vec{r}}\cdot\vec{r} = 0.$$

Example: Write Liebniz Product Rule for Cross Product:

Solution:

$$\frac{d}{dt}\left(\vec{r}_1(t)\times\vec{r}_2(t)\right) = \dot{\vec{r}}_1(t)\times\vec{r}_2(t) + \vec{r}_1(t)\times\dot{\vec{r}}_2(t)$$

"Whenever its called a product, Liebniz Product Rule holds!"